ENERGY THEOREMS IN STRUCTURAL MECHANICS

by

A. Jennings*

1. *Introduction*

Various theorems of strain energy, complementary energy, potential energy and virtual work have been used extensively in the field of structural analysis. Attempts which have been made to collect and relate all the various theorems [1-5] tend to show that the field of application of some of the theorems overlap. In cases where deflection or deflection compatibility relationships are required then the use of complementary energy [6] ensures a wider range of application than the use of theorems involving strain energy because the first theorem of complementary energy is applicable in cases where nonlinearities exist in the material stress strain characteristics.

When considering structures in which the deformations are not small, then none of the established energy theorems used in the conventional way will yield deflection or deflection compatibility equations. The origin of this weakness stems from the nature of complementary energy, for, unlike strain energy, it is not a form of potential energy. Westergaard [7] and Charlton have shown that complementary energy is not conserved for structures undergoing gross deformation. However Libove [8], Charlton [9] and Levinson [10] have shown that it is possible to develope new theorems or adapt existing ones to cater for the case of gross deformation. Although achieving the same result their approach is entirely different. Libove defines a new form of complementary energy and developes a theorem of stationary total complementary energy which is contrasted with the stationary principle of total potential energy, whereas Charlton uses the principle of virtual work in an unconventional way. *It would appear necessary that any new energy theorems or principles should be discussed in as many ways as possible before they can take their proper place alongside the established theorems.* The present paper is continuing this discussion by considering the following questions:

a) It is really necessary to increase the scope of the energy theorems?
b) What is the simplest form to present any new theorems?
c) How should such theorems be justified?
d) What is their relationship to the existing theorems?

In order to discuss the necessity of increasing the scope of energy theorems some equilibrium and compatibility equations are developed which can arise in the analysis of a freely hanging cable.

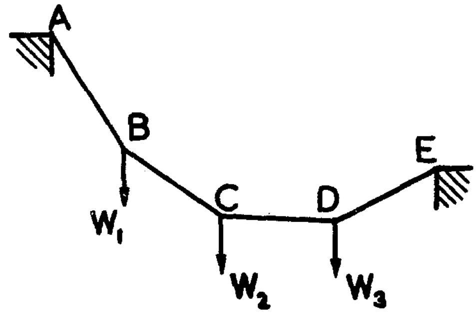

Fig. 1. A freely hanging cable.

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2. The Free Cable

The freely hanging cable can be analysed by the force method allowing for gross deformations [11,12]. Consider the cable shown in Fig. 1, which hangs between two supports A and E and carries three vertical loads W_1, W_2 and W_3 at B, C and D respectively. Assuming that the cable has no bending stiffness it will adopt a series of straight lines between the points of load application. Its behaviour is similar to a chain of bars and when completely unloaded its position in space is undefined. However, when a load has been applied to the cable its subsequent characteristics are similar to that of a structure, in that it deflects finite amounts under additional load, it can store energy and return to its original equilibrium position when the additional load is removed. A significant problem would be that of determining the position of the cable for a given set of loads. An outline of a force method analysis of the cable will now be given.

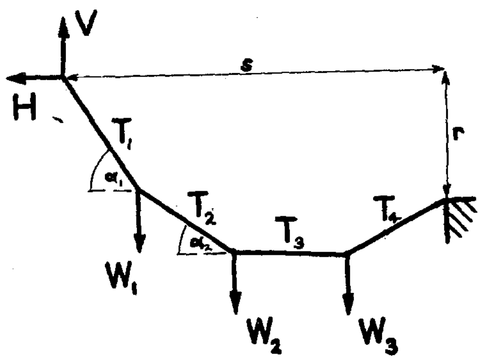

Fig. 2. Statically determinate system corresponding to free cable.

Assume numerical values for H and V, the horizontal and vertical components of the support reaction at A. Then the cable as shown in Fig. 2 is statically determinate which means that the tension and slope of all the cable links can be obtained by considering only statical equilibrium, giving

$$\begin{aligned} T_1^2 &= H^2 + V^2, \\ \cos\alpha_1 &= H/T_1, \quad \sin\alpha_1 = V/T_1, \\ T_2^2 &= H^2 + (V - W_1)^2, \\ \cos\alpha_2 &= H/T_2, \quad \sin\alpha_2 = (V - W_1)/T_2, \\ &\text{etc.} \end{aligned} \tag{1}$$

The strained lengths of the cable links can then be obtained and the span s and rise r of the cable can be obtained from the following compatibility conditions

$$\begin{aligned} s &= l_1' \cos\alpha_1 + l_2' \cos\alpha_2 + l_3' \cos\alpha_3 + l_4' \cos\alpha_4 \\ r &= l_1' \sin\alpha_1 + l_2' \sin\alpha_2 + l_3' \sin\alpha_3 + l_4' \sin\alpha_4 \end{aligned} \tag{2}$$

where l_i' is the strained length of cable link i.

In general the calculated span and rise will not be correct and it is necessary to adjust H and V in such a way as to correct s and r. A rapid procedure for adjusting H and V is to develope the linear flexibility equations which define the changes in span and rise, δs and δr, in terms of changes in the support reactions δH and δV. By putting δs and δr equal to the required correction then linear estimates of the correct support reactions can be obtained. Errors arise through the nonlinear behaviour of the cable and hence iteration is required until a suitable accuracy is achieved.

One method of determining the flexibility equations associated with incremental changes in the support reactions is to, first of all, obtain the corresponding change in the cable link tensions. From eq. 1 it is seen that

$$\left.\begin{array}{lrl} & T_1 \delta T_1 & = H \delta H + V \delta V \\ \text{giving} & \delta T_1 & = \cos \alpha_1 \, \delta H + \sin \alpha_1 \, \delta V \\ \text{similarly} & \delta T_2 & = \cos \alpha_2 \, \delta H + \sin \alpha_2 \, \delta V \\ \text{etc.} & & \end{array}\right\} \qquad (3)$$

Then it is necessary to determine the change in slope of the cable links and from these to deduce the change in span and rise of the entire cable.

It is not the object of the paper to formulate the solution to the free cable problem, the reason for extending the analysis as far as this is to show the significance of eqs. (2) and (3). These particular equations, when written in matrix form, show an obvious relationship with each other. Eq. (3) is

$$\begin{bmatrix} \delta T_1 \\ \delta T_2 \\ \delta T_3 \\ \delta T_4 \end{bmatrix} = \begin{bmatrix} \cos \alpha_1 & \sin \alpha_1 \\ \cos \alpha_2 & \sin \alpha_2 \\ \cos \alpha_3 & \sin \alpha_3 \\ \cos \alpha_4 & \sin \alpha_4 \end{bmatrix} \begin{bmatrix} \delta H \\ \delta V \end{bmatrix} \qquad (4)$$

and eq. (2) is

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} \cos \alpha_1 & \cos \alpha_2 & \cos \alpha_3 & \cos \alpha_4 \\ \sin \alpha_1 & \sin \alpha_2 & \sin \alpha_3 & \sin \alpha_4 \end{bmatrix} \begin{bmatrix} l_1' \\ l_2' \\ l_3' \\ l_4' \end{bmatrix} \qquad (5)$$

This correspondence of these equations could be explained as coincidence, however similar results are obtained for other cable configurations and also for pin-jointed frame problems in which gross deformation is taken into consideration.

Transposed relationships between the matrices involved in equilibrium and compatibility equations are a well known feature of matrix structural analysis, but in other cases the reason for this correspondence can be justified by reference to appropriate energy theorems. However in the case under consideration none of the widely accepted energy theorems is applicable. Because incremental changes in the force system are involved, all but complementary energy theorems must be excluded. But the complementary energy theorems must be excluded. But the complementary energy for the cable cannot be obtained because the unloaded position of the cable is indeterminate and due to gross deformation the first theorem of complementary is not valid.

3. The Relevance of New Energy Theorems

It would be of advantage if the structural analyst were to be aware of any relationship between the equilibrium and compatibility equations which arise in the particular problem under consideration. Such relationships could then be used either to save time in the analysis or as a partial check on the equations used. However, unless any underlying energy theorems are properly enunciated then it would not be possible to forsee the situations in which these equations would correspond. In cases where approximations are made about the force system to produce a small number of variables in an analysis, the need to develope energy theorems to cater for the case of gross deformation is even more pronounced.

It may be true that for most structures undergoing gross deformation it is simpler to adopt a displacement method analysis. However the problem of the free cable tends to suggest that this may not always be so. The number of unknown forces for the free cable is always two no matter how many vertical loads are applied to the cable, and therefore, a force method solution would seem to be particularly suitable.

The name 'gross deformation' may be misleading to the extent of giving the impression that it would only apply to deformations beyond those usually tolerable in real structures. Whereas this might normally be true, it is necessary to use gross deformation theory whenever the equilibrium equations for the structure are affected by its deflection. Thus it may be relevant to cable and suspension structures and also to structures near to the buckling condition.

4. *Basic Energy Theorems*

Of the accepted energy theorems in structural mechanics the theorem of minimum total potential energy and the principle of virtual work appear to be the most fundamental. The theorem of minimum total potential energy follows from the principle of conservation of energy. If energy is not converted into heat by internal or external friction then small movements of a structure or mechanism will tend to result in interchange entirely between potential and kinetic energy. The system will only remain in stable equilibrium if all possible small movements result in an increase in potential energy. This theorem hardly needs any elaborate justification as simple illustrations such as the 'ball in bowl' suffice.

Application of the theorem of minimum total potential energy to structures involves introducing the concept of strain energy to describe the work done on the structure on account of the movement of the applied forces already on the structure as straining takes place.

The principle of virtual work for a conservative system is equivalent to the stationary part of the theorem of minimum total potential energy, and the normal justification would similarly be by reasoning that any virtual displacement which causes a decrease in potential energy must imply an increase in kinetic energy of the system and therefore cannot occur if the system is in equilibrium. However the implication that the principle of virtual work is just an alternative way of stating part of the theorem of minimum total potential energy is not completely true firstly because the principle was known before the principle of conservation of energy [13] and secondly because it can be successfully applied to nonconservative systems. Virtual work can be applied to the case of a block at rest on a rough inclined plane. The forces on the block are assumed to remain constant over any virtual displacement of the block irrespective of whether the forces would remain constant over an actual displacement however small, or even whether such an actual displacement could take place. Thus a virtual displacement of the forces in a direction normal to the inclined plane would be acceptable even though the block could not make this movement. A justification of the use of virtual work for nonconservative systems would be that the forces are assumed to be conservative over the virtual displacement, and the fact that the actual forces do not behave in the same way does not affect the actual equilibrium conditions deduced.

5. *A Virtual Work Principle for Incremental Forces*

The principle of virtual work can be written as follows: *If a body or system of bodies is in equilibrium and is given an arbitrary incremental displacement, the work done by the external forces acting on the system in this displacement is zero.* In the interpretation of this principle it is understood that only displacements are considered in which the various parts of the system remain compatible with each other.

An alternative virtual work principle involving changes in the forces rather than changes in displacement could be stated as follows: *If a system of bodies is such that the position of all the bodies are mutually compatible, than for any arbitrary incremental change in the forces consistent with equilibrium the work required to apply the incremental forces to the system is zero.*

Thus the roles of compatibility of displacements and equilibrium of forces have been interchanged.

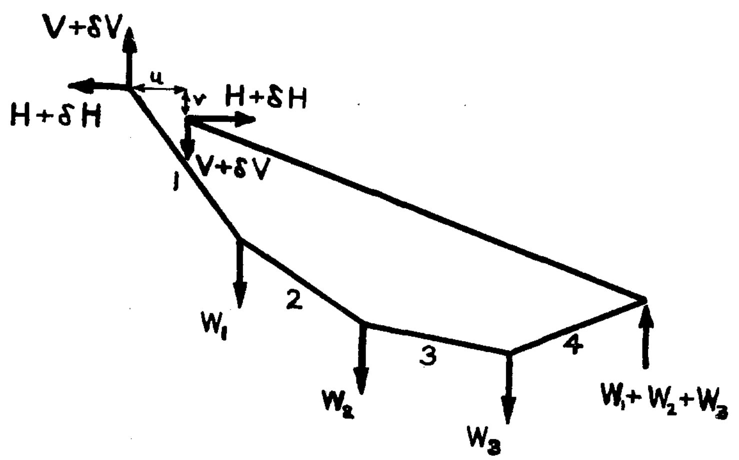

Fig. 3. Five bar chain corresponding to free cable.

In order to apply this principle to the cable problem it would be necessary to include the supports in the system. The cable and support system could be considered as the five bar chain shown in Fig. 3 in which the fifth bar is rigid and held firmly in position. Suppose that there is a lack of fit at the left hand end of the cable involving displacement components u horizontally and v vertically. Then if incremental changes in the forces δH and δV are made the work done in applying these increments is equal to

$$\delta \overline{W} = -u\delta H - v\delta V \tag{6}$$

The negative values arise because separation of the incremental forces to the positions as shown in Fig. 3 involves a work output by the forces rather than work being done on them. If u and v are zero then the work done in applying the incremental forces δH or δV to the system are both zero. Because the cut system is statically determinate all the possible variations in the forces consistent with equilibrium have been considered and therefore the principle is upheld in this case.

A further example in which the principle can be seen to be upheld is the doubly symmetrical pin-jointed frame shown in Fig. 4

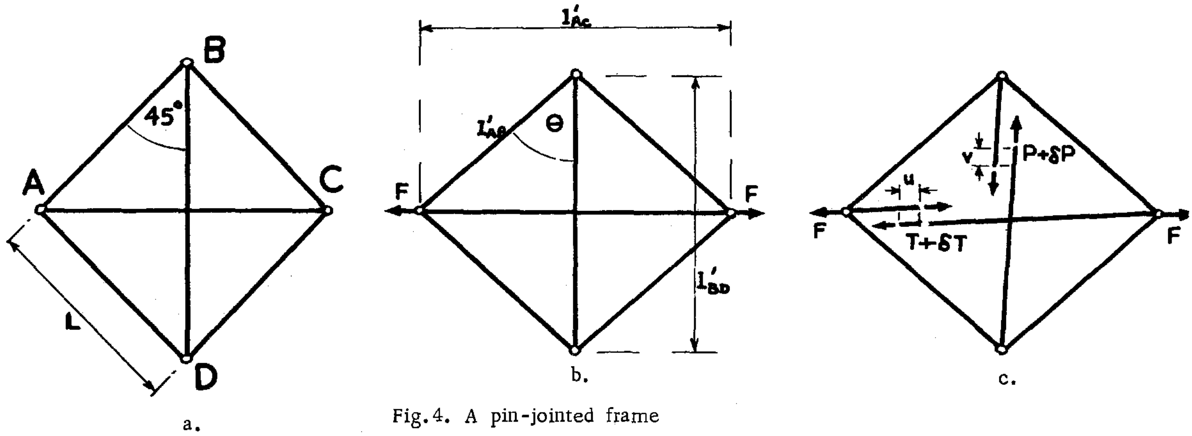

Fig. 4. A pin-jointed frame
a) unloaded
b) loaded
c) statically determinate configuration.

which is assumed to undergo gross deformation. Because gross deformation is taken into account the forces in the members can only be obtained by means of joint equilibrium when two forces are known. A force method analysis of the frame could be achieved by considering the members AC and BD cut with forces of T and P acting ross the cuts, and then adjusting the values of T and P until lack of fit across the cuts have been eliminated. If overlap occurs in the cut members to the extent of u in AC and v in BD then the work done in applying the increments δT and δP to the forces acting across the cuts is

$$\delta \overline{W} = -u\,\delta T - v\,\delta P \tag{7}$$

As the only possible changes in the force system consistent with equilibrium involve changes in T or P and the work expression is zero when u = v = 0 then the principle is again upheld.

6. Application of the Virtual Work Principle for Incremental Forces.

In order to use this principle it is necessary to obtain the work done in applying the incremental forces system by summing the work done in applying the incremental forces to the individual bodies in the system. One method of evaluating the work done in applying incremental forces to pin-ended members is to consider the horizontal and vertical components of the incremental forces as shown for member i in Fig. 5. If the incremental

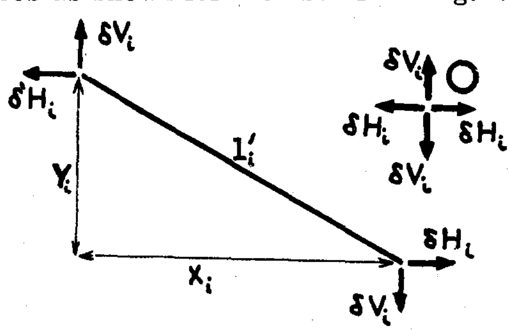

Fig. 5. Pin-ended bar with horizontal and vertical force increments.

forces are moved onto the member from any arbitrary origin, say 0 in Fig. 5, and if X_i and Y_i are the horizontal and vertical projections of the member strained length then the work done by the incremental forces δH_i and δV_i during application is

$$\delta \overline{W}_i = - X_i \delta H_i - Y_i \delta V_i \tag{8}$$

In the case of the five bar chain (Fig. 3), with link incremental forces defined in Fig. 6 then from equilibrium

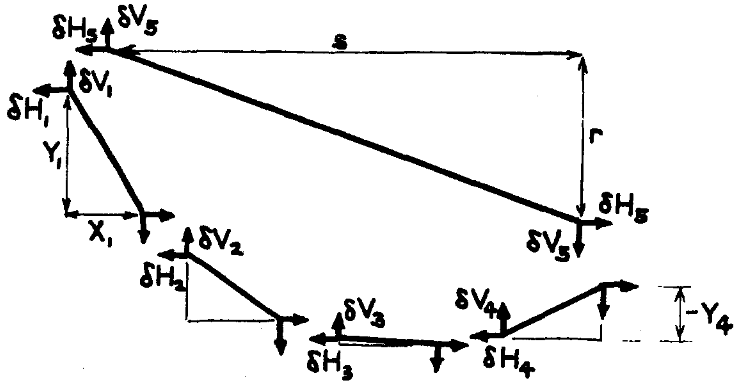

Fig. 6. Five bar chain with horizontal and vertical force increments

$$\left.\begin{aligned} &\delta H_1 = \delta H_2 = \delta H_3 = \delta H_4 = -\delta H_5 = \delta H \\ \text{and}\quad &\delta V_1 = \delta V_2 = \delta V_3 = \delta V_4 = -\delta V_5 = \delta V \end{aligned}\right\} \tag{9}$$

Hence in applying the incremental force δH the work done is

$$\delta \overline{W} = -(X_1 + X_2 + X_3 + X_4 - s)\delta H = 0 \tag{10}$$

which gives the compatibility condition

$$X_1 + X_2 + X_3 + X_4 = s \tag{11}$$

and in applying the incremental force δV the work done is

$$\delta \overline{W} = -(Y_1 + Y_2 + Y_3 + Y_4 - r)\delta V = 0 \tag{12}$$

which gives the compatibility condition

$$Y_1 + Y_2 + Y_3 + Y_4 = r \tag{13}$$

Equations (11) and (13) are the only equations needed to ensure compatibility of the system.

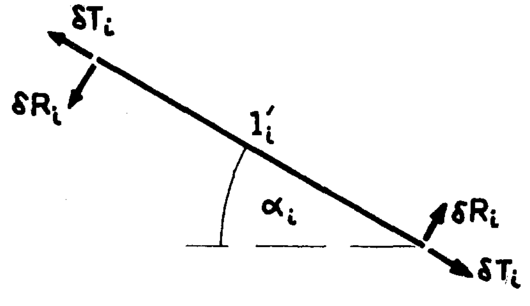

Fig. 7. Pin-ended bar with axial force increments

An alternative way of evaluating the work done in applying incremental forces to pin-ended members is to consider components parallel and perpendicular to the bar axis (Fig. 7). The components of force δR_i which act perpendicular to the bar axis are present to take account of possible changes in the inclination of the bar. However they can be applied without doing work as the forces only need to be moved perpendicular to their line of action. Hence the work done in applying the incremental forces

$$\delta \overline{W}_i = l'_i \delta T_i \tag{14}$$

and for the five bar chain

$$-(l'_1 \delta T_1 + l'_2 \delta T_2 + l'_3 \delta T_3 + l'_4 \delta T_4) + s\delta H + r\delta V = 0 \tag{15}$$

which in matrix form is

$$\begin{bmatrix} s & r \end{bmatrix} \begin{bmatrix} \delta H \\ \delta V \end{bmatrix} = \begin{bmatrix} l'_1 & l'_2 & l'_3 & l'_4 \end{bmatrix} \begin{bmatrix} \delta T_1 \\ \delta T_2 \\ \delta T_3 \\ \delta T_4 \end{bmatrix} \tag{16}$$

Substituting for $\{\delta T_1 \; \delta T_2 \; \delta T_3 \; \delta T_4\}$ from equation 4 gives

$$\begin{bmatrix} s & r \end{bmatrix} \begin{bmatrix} \delta H \\ \delta V \end{bmatrix} = \begin{bmatrix} l'_1 & l'_2 & l'_3 & l'_4 \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \\ \cos\alpha_3 & \sin\alpha_3 \\ \cos\alpha_4 & \sin\alpha_4 \end{bmatrix} \begin{bmatrix} \delta H \\ \delta V \end{bmatrix} \tag{17}$$

As this equation is valid for δH or δV applied separately then $\{\delta H \; \delta V\}$ may be cancelled and on transposing equation 5 is obtained.

Considering the pin-jointed frame of Fig. 4, the application of the virtual work principle for incremental force gives

$$-4l'_{AB}\delta T_{AB} - l'_{AC}\delta T - l'_{BD}\delta P = 0 \tag{18}$$

But from joint equilibrium

$$\delta T_{AB} = -\tfrac{1}{2}\sin\theta \, \delta T - \tfrac{1}{2}\cos\theta \, \delta P \tag{19}$$

Hence

$$(-2l'_{AB}\sin\theta + l'_{AC})\delta T + (-2l'_{AB}\cos\theta + l'_{BD})\delta P = 0 \tag{20}$$

As this equation is true for increments δT and δP applied separately

$$\left.\begin{aligned} l'_{AC} &= 2l'_{AB}\sin\theta \\ \text{and } l'_{BD} &= 2l'_{AB}\cos\theta \end{aligned}\right\} \tag{21}$$

These compatibility equations can be seen to be valid by inspection of Fig. 4b. What is more they are the only compatibility equations necessary to ensure that the frame will fit together in its deformed state.

When using the principle of virtual work for incremental displacements it is usual to assume that by satisfying the virtual work principle for all possible virtual displacements a sufficient as well as necessary condition for equilibrium has been obtained. When considering the principle of virtual work for incremental forces then any incompatible state involves an unclosedloop in the system around which may be specified a set of incremental forces in equilibrium. Hence there must be a possible variation in the force system for which the work done in applying the forces will not be zero. The corollary of this is that if the principle has been upheld for all possible variations in the force system then compatibility is assured. As with the conventional virtual work principle this new principle is applicable whether or not the system is conservative.

7. Stress Energy

Because energy is the capacity for doing work there is likely to be an energy theorem which has the same effect as the principle of virtual work for incremental forces. In order to develope such a theorem it is useful to introduce a new energy form which could be called *stress energy*, and is defined as the total work done in applying the force system to the structure or mechanism. It is assumed that the forces are transferred from any arbitrary origin to the structure without any change of magnitude or direction and at a vanishingly slow rate so that no kinetic energy is developed. For a conservative system it must be possible at any time to restore all the loads to the origin without any overall net gain or loss in energy and hence the work done in applying the forces to the structure must be recoverable. It is possible for the stress energy to be negative, implying that work is done by the forces in being transferred from the origin onto the structure, this work being required again if the forces are to be moved back to the origin.

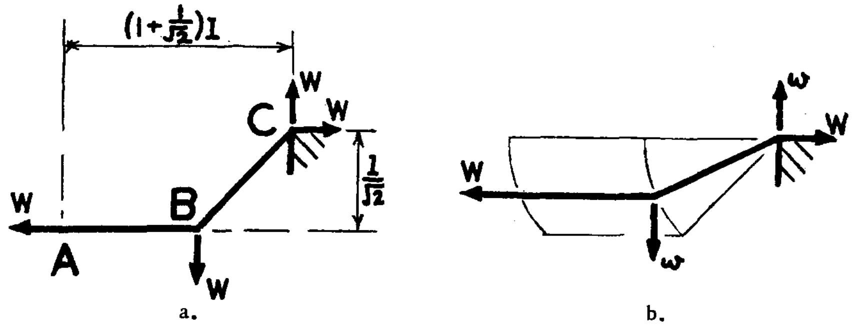

Fig. 8. A two bar mechanism
a) fully loaded
b) partially loaded

The uniqueness of the stress energy function can be illustrated by considering the two bar mechanism ABC shown in Fig. 8a. Consider the case where the bars are inextensible of length l, and the stress energy is required for the two loads shown. The loads may be applied simultaneously at the same rate in which case the geometry of the bars will remain unaltered and the stress energy will be

$$S = -\left(1 + \frac{1}{\sqrt{2}}\right) Wl - \frac{1}{\sqrt{2}} Wl = -(1 + \sqrt{2}) Wl \tag{22}$$

If, on the other nand, the horizontal load is applied before the vertical load, the geometry of the system will change during loading. When the horizontal load is being applied the bars will both be horizontal and therefore the stress energy of the system after all the load at A has been applied will be -2Wl. Considering now the situation when load w has been applied at B in addition to the full load at A, the vertical projection of BC will be $\frac{wl}{(W^2 + w^2)^{\frac{1}{2}}}$ (Fig. 8b) and the addition to the stress energy due to an addition dw will be

$$dS = \frac{-wldw}{(W^2 + w^2)^{\frac{1}{2}}} \tag{23}$$

Hence the total stress energy is

$$S = -2Wl - \int_o^w \frac{wl\,dw}{(W^2 + w^2)^{\frac{1}{2}}} = -(1 + \sqrt{2})\ Wl \tag{24}$$

which corresponds to the result obtained when the loads were applied simultaneously. If the bars are not rigid, but linear elastic with Youngs modulus E and cross-sectional area A, then whichever way the two loads are applied the stress energy will be

$$S = -(1 + \sqrt{2})\ Wl - \frac{3W^2l}{2EA} \tag{25}$$

If there is conservation of energy then the stress energy of a system of bodies must be equal to the sum of the stress energies of the component bodies. Therefore a method of evaluating the stress energy of a structure is to sum the components arising from the members.

In the case of a pin-ended bar of unstrained length l and extension e under load t, the increment to the stress energy arising from an increment to t is

$$dS = -(l + e)dt \tag{26}$$

Therefore if t is increased from zero to T the stress energy for the bar under the load T is given by

$$S = -\int_{t=0}^{T} (l + e)dt \tag{27}$$

For a bar in tension having nonlinear stress-strain characteristics the stress energy could be represented by minus the area ABCD in Fig. 9a. For a nonlinear bar in compression the stress energy could be represented by the Area ABCD in Fig. 9b. If the bar has linear stress-strain characteristics with Youngs modulus E and cross-sectional area A then

$$S = -Tl - \frac{T^2l}{2EA} \tag{28}$$

Both equations 27 and 28 are valid for compression members as well as tension members as long as T and e adopt negative values for compression.

In the example of Fig. 8 the stress energy of the bars allowing for elasticity are given by

$$\left.\begin{aligned} S_{AB} &= -Wl - \frac{W^2l}{2EA} \\ S_{BC} &= -2Wl - \frac{W^2l}{EA} \end{aligned}\right\} \tag{29}$$

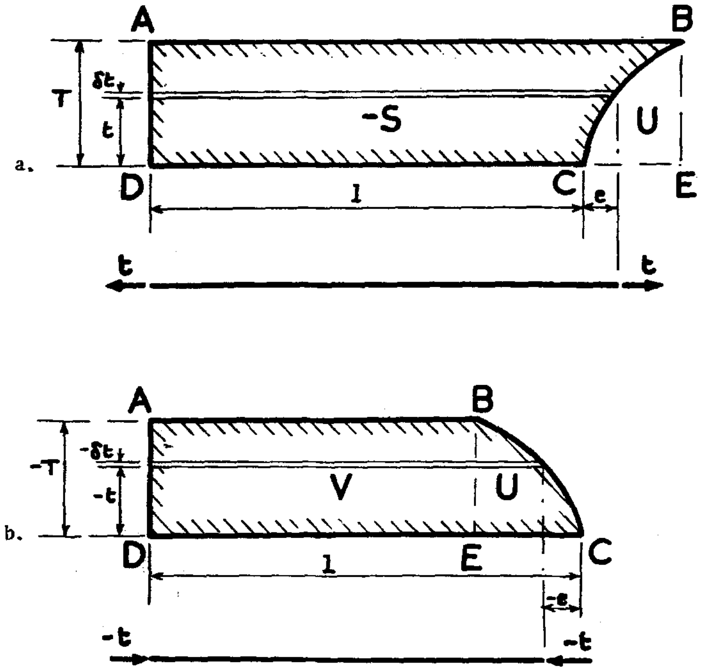

Fig. 9. Stress energy for a pin-ended bar with nonlinear load deflection characteristics
a) in tension
b) in compression

Summing these stress energies gives the total stress energy which agrees with equation 25.

8. *A Theorem of Stress Energy*

The stress energy of a structure or mechanism can always be written as a function of a set of independent forces. In the case of structures or mechanisms which are statically indeterminate when undergoing gross deformation some of the independent forces will be unknowns. The state of the free cable of Fig. 1 is entirely defined by the forces H, V, W_1, W_2, and W_3, hence the stress energy could be specified in terms of these forces. Similarly the stress energy of the pin-jointed frame of Fig. 4 could be obtained as a function of F, T and P.

Each independent force together with the corresponding reaction and internal forces could be considered as an equilibrating force system. Where the applied forces are internal forces (e.g. T or P of Fig. 4) introduced at imaginary cuts in the structure then the force system could be considered as self-equilibrating. It is to be noted however, that the force systems do not appear to be equilibrating so far as moment equilibrium is concerned. This apparent anomaly arises because the line of action of the forces are not fixed but automatically adjust in such a way that all moment equilibrium equations are satisfied.

Considering the stress energy of a structure or mechanism as a function of n independent forces than

$$S = S(F_1 \;.....\; F_j \;.....\; F_n) \qquad (30)$$

If infinitely small increments are added to the forces then the increment to the stress energy is

$$\delta S = \sum_{j=1}^{n} \nabla_j \delta F_j$$

where ∇_j is the separation between the force F_j and its reaction. But from equation 30

$$\delta S = \sum_{j=1}^{n} \frac{\partial S}{\partial F_j} \delta F_j \tag{32}$$

As equations 31 and 32 must be true for any possible variation in the force system then

$$\frac{\partial S}{\partial F_j} = \nabla_j \tag{33}$$

for any value of j.

This could be stated as the following theorem:

The differential coefficient of the stress energy with respect to a force is the distance, measured parallel to the line of action of the force, which separates the force and its reaction.

In this definition it is implied that a positive separation is one in which the force and its reaction have a positive potential energy.

In the case of a pin-jointed frame having members then

$$S = \sum_{i=1}^{m} S_i \tag{34}$$

where S_i is the stress energy of member i.

But S_i is a function of T_i only, hence

$$\nabla = \sum_{i=1}^{m} \frac{dS_i}{dT_i} \frac{\partial T_i}{\partial F_j} \tag{35}$$

$$\text{As } \frac{dS_i}{dT_i} = -l_i' \tag{36}$$

where l_i' is the strained length of the member

$$\nabla_j = - \sum_{i=1}^{m} l_i' \frac{\partial T_i}{\partial F_j} \tag{37}$$

Consider the cable of Fig. 2 with F_j set to H and V in turn, then equation 37 gives

$$s = - \sum_{i=1}^{4} l_i' \frac{\partial T_i}{\partial H}, \quad r = - \sum_{i=1}^{4} l_i' \frac{\partial T_i}{\partial V} \tag{38}$$

Using the equilibrium equation 3 to obtain $\frac{\partial T_i}{\partial V}$ and $\frac{\partial T_i}{\partial V}$ gives

$$s = - \sum_{i=1}^{4} l_i' \cos \alpha_i \quad , \quad r = - \sum_{i=1}^{4} l_i' \sin \alpha_i \tag{39}$$

which is an alternative form for the compatibility equations 2.

Considering the pin-jointed frame of Fig. 4, then if ∇_T and ∇_P are the overlaps at the cuts in members AC and BD respectively, equation 37 gives

$$\left.\begin{aligned}\nabla_T &= -\,4l'_{AB}\frac{\partial T_{AB}}{\partial T} - l'_{AD}\frac{\partial T_{AD}}{\partial T} - l'_{BC}\frac{\partial T_{BC}}{\partial T}\\ \text{and }\nabla_P &= -\,4l'_{AB}\frac{\partial T_{AB}}{\partial P} - l'_{AD}\frac{\partial T_{AD}}{\partial P} - l'_{BC}\frac{\partial T_{BC}}{\partial P}\end{aligned}\right\}$$

Using the equilibrium equation 19 and putting ∇_T and ∇_P equal to zero gives

$$\left.\begin{aligned}\nabla_T &= 2l'_{AB}\sin\theta - l'_{AD} = 0\\ \text{and }\nabla_P &= 2l'_{AB}\cos\theta - l'_{BD} = 0\end{aligned}\right\}$$

which is a statement of the compatibility equations previously obtained by using the virtual work principle for incremental forces.

Thus the stress energy theorem can be used to obtain any of the results previously obtained by the principle of virtual work for incremental forces and it can be seen that by applying ficticious forces any projected distance can be obtained. For example if the projection ∇ of the distance BC onto a line with an inclination of 45° is required for the cable problem, then the stress energy may be differentiated with respect to the pair of opposing forces P shown in Fig. 10.

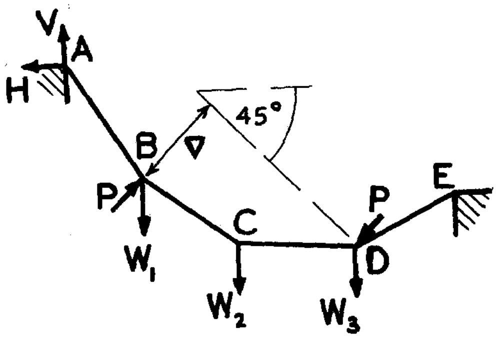

Fig. 10. Free cable with additional imaginary load.

9. *Application to a Suspension Bridge Structure*

Consider the simple suspension bridge structure shown in Fig. 11, and let it be assumed that
a) the deck girder is horizontal when it has not load acting on it,
b) small deflection theory is adequate for the deck girder,
c) the effect of hanger inclination can be ignored.

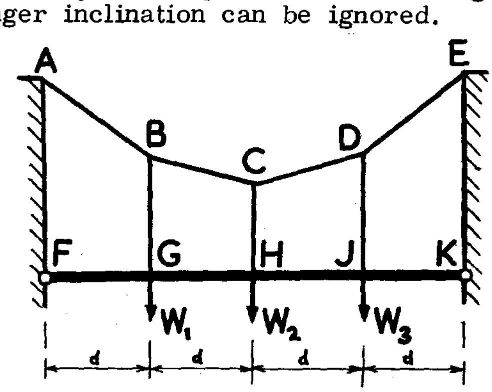

Fig. 11. A simple suspension bridge.

This structure can be made statically determinate by cutting the cable at the left hand support and inserting pin-joints into the girder at G, H and J. By assuming that F is supported from A by a rigid wire and by introducing redundant cable reactions H and V and redundant girder moments M_1, M_2, and M_3, the structure shown in Fig. 12 is obtained. If θ_1, θ_2 and θ_3 are the rotations of the hinges corresponding to M_1, M_2, and

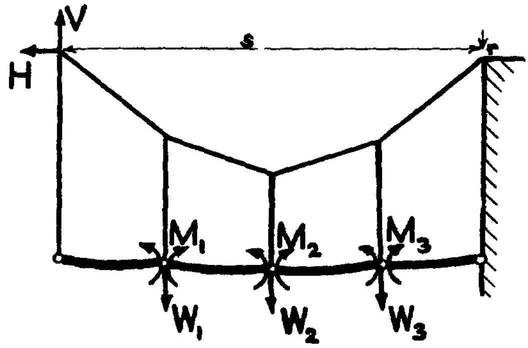

Fig. 12. Statically determinate system for suspension bridge.

M_3, then the problem of analysis is to determine the values of the redundants such that the span and rise of the cable are correct and $\theta_1 = \theta_2 = \theta_3 = 0$.

The stress energy for the structure may be evaluated. However it is simpler to evaluate directly the increment in the stress energy for incremental changes in the forces.

As couples are included in the force system it is necessary to determine the increment in stress energy arising from an increment to a couple. A couple M can be represented by a pair of equal and opposite forces P acting at a distance y apart. If there is a small rotation θ then with the direction of the forces P unaltered, the work done in applying an increment to the forces is - ydP (see Fig. 13). Hence the increment in stress energy

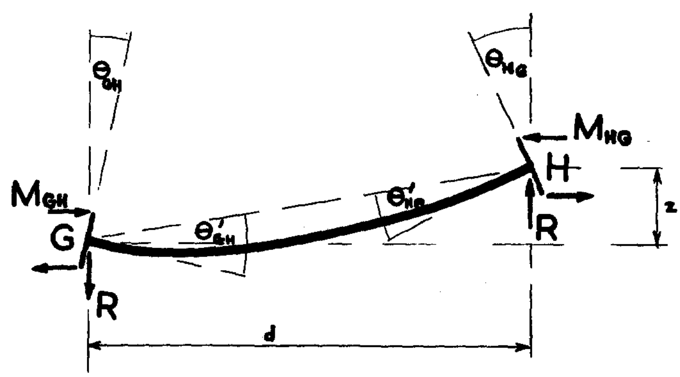

Fig. 13. Loaded configuration of suspension bridge deck girder member.

for an increment to a couple M is

$$\delta S = -\theta\,\delta M \tag{42}$$

If θ_1, θ_2 and θ_3 are the rotations of the hinges 1, 2 and 3, the increment to the stress energy due to increments in all the redundants forces is

$$\delta S = -s\delta H - r\delta V - \theta_1\delta M_1 - \theta_2\delta M_2 - \theta_3\delta M_3 \tag{43}$$

In order to evaluate the increment to the stress energy by summing the contributions of the individual members it is necessary to have an expression for the increment to the stress energy of a typical deck segment. Consider the segment GH in which the end moments are M_{GH} and M_{HG}, the end rotations are θ_{GH} and θ_{HG}, the sway is z and the shear force is R as shown in Fig. 13. The stress energy due to increments to the applied forces is

$$\delta S = -\theta_{GH}\,\delta M_{GH} - \theta_{HG}\,\delta M_{HG} - z\delta R \tag{44}$$

But from equilibrium

$$\delta R = \frac{1}{d}\left(\delta M_{GH} - \delta M_{HG}\right) \tag{45}$$

Hence $$\delta S = -\left(\theta_{GH} + \frac{z}{d}\right)\delta M_{GH} - \left(\theta_{HG} - \frac{z}{d}\right)\delta M_{HG} \tag{46}$$

Defining θ'_{GH} and θ'_{HG} as the end rotations measured relative to the line GH then

$$\delta S = -\theta'_{GH}\delta M_{GH} - \theta'_{HG}\delta M_{HG} \tag{47}$$

which is equal to minus the change in complementary energy due to a change in the end moments.

The total stress energy obtained by summing all the member contributions is thus

$$\delta S = -\Sigma l'_e\,\delta T_e - \Sigma\theta'_f\delta M_f \tag{48}$$

where the first summation is for e = AB, BC, CD, DE, AF, BG, CH, and DJ and the second summation is for f = GF, GH, HG, HJ, JH, and JK.

The hanger forces can be obtained by considering the equilibrium of the various deck segments of Fig. 12

$$\left.\begin{aligned} T_{AF} &= M_1/d \\ T_{BG} &= W_1 + (-2M_1 + M_2)/d \\ T_{CH} &= W_2 + (M_1 - 2M_2 + M_3)/d \\ T_{DJ} &= W_3 + (M_2 - 2M_3)/d \end{aligned}\right\} \tag{49}$$

and the cable tensions can be obtained from the equilibrium of the cable

$$\left.\begin{aligned} T^2_{AB} &= H^2 + [V - M_1/d]^2 \\ T^2_{BC} &= H^2 + [V - W_1 + (M_1 - M_2)/d]^2 \\ T^2_{CD} &= H^2 + [V - W_1 - W_2 + (M_2 - M_3)/a]^2 \\ T^2_{DE} &= H^2 + [V - W_1 - W_2 - W_3 + M_3/d]^2 \end{aligned}\right\} \tag{50}$$

Hence the following equilibrium forces for incremental forces may be obtained:

$$
\begin{bmatrix} \delta T_{AB} \\ \delta T_{BC} \\ \delta T_{CD} \\ \delta T_{DE} \\ \cdots \\ \delta T_{AF} \\ \delta T_{BG} \\ \delta T_{CH} \\ \delta T_{DJ} \\ \cdots \\ \delta M_{GF} \\ \delta M_{GH} \\ \delta M_{HG} \\ \delta M_{HJ} \\ \delta M_{JH} \\ \delta M_{JK} \end{bmatrix}
=
\begin{bmatrix}
\cos\alpha_1 & \sin\alpha_1 & \cdot & -\sin\alpha_1/d & 0 & 0 \\
\cos\alpha_2 & \sin\alpha_2 & \cdot & \sin\alpha_2/d & -\sin\alpha_2/d & 0 \\
\cos\alpha_3 & \sin\alpha_3 & \cdot & 0 & \sin\alpha_3/d & -\sin\alpha_3/d \\
\cos\alpha_4 & \sin\alpha_4 & \cdot & 0 & 0 & \sin\alpha_4/d \\
\cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\
0 & 0 & \cdot & 1/d & 0 & 0 \\
0 & 0 & \cdot & -2/d & 1/d & 0 \\
0 & 0 & \cdot & 1/d & -2/d & 1/d \\
0 & 0 & \cdot & 0 & 1/d & -2/d \\
\cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\
0 & 0 & \cdot & 1 & 0 & 0 \\
0 & 0 & \cdot & 1 & 0 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 \\
0 & 0 & \cdot & 0 & 0 & 1 \\
0 & 0 & \cdot & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \delta H \\ \delta V \\ \cdots \\ \delta M_1 \\ \delta M_2 \\ \delta M_3 \end{bmatrix}
\delta M_4
\qquad (51)
$$

By substituting the values of the incremental forces fromeequation 51 into equation 48 and equating the increment in stress energy to that of equation 43, it follows that

$$
\begin{bmatrix} s \\ r \\ \cdots \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}
=
$$

$$
\left[\begin{array}{cccc|cccc|cccccc}
\cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 & \cos\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 & \sin\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & & \cdots & \cdots & & \cdots & & & \cdots & & & & \\
-\sin\alpha_1/d & \sin\alpha_2/d & 0 & 0 & 1/d & -2/d & 1/d & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -\sin\alpha_2/d & \sin\alpha_3/d & 0 & 0 & 1/d & -2/d & 1/d & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -\sin\alpha_3/d & \sin\alpha_4/d & 0 & 0 & 1/d & -2/d & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{bmatrix} 1'_{AB} \\ 1'_{BC} \\ 1'_{CD} \\ 1'_{DE} \\ \cdots \\ 1'_{AF} \\ 1'_{BG} \\ 1'_{CH} \\ 1'_{DJ} \\ \cdots \\ \theta'_{GF} \\ \theta'_{GH} \\ \theta'_{HG} \\ \theta'_{HJ} \\ \theta'_{JH} \\ \theta'_{JK} \end{bmatrix}
\qquad (52)
$$

The equations for s and r agree with the previously derived free cable equations. A typical equation for hinge rotation is

$$
\theta = (1'_{AF} - 21'_{BG} + 1'_{CH} - 1'_{AB}\sin\alpha_1 - 1'_{BC}\sin\alpha_2)/d + \theta'_{GF} + \theta'_{GH} \qquad (53)
$$

the validity of which can be checked by referring to Fig. 14.

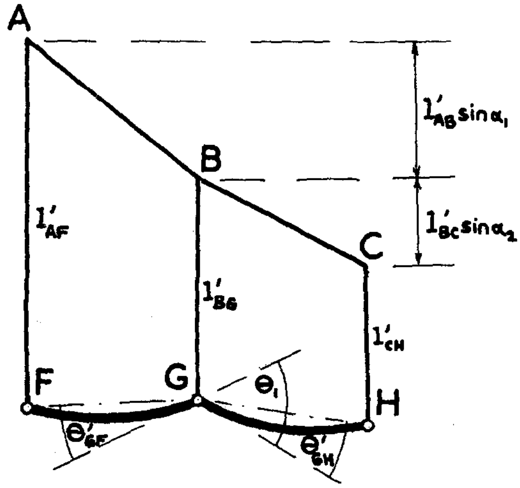

Fig. 14. Displaced configuration for two panels of suspension bridge.

10. Physical Significance of Stress Energy

The stress energy of a conservative structure is the total energy which can be recovered by returning the loads back to an arbitrary origin. This energy is stored in the form of potential energy of the applied forces V and strain energy U, hence

$$S = V + U \tag{54}$$

The stress energy is therefore the total potential energy of the system consisting of the structure or mechanism and the loads acting on it, and hence could be described by the alternative same system energy. Equation 54 can easily be verified for the case of the pin-ended bar in tension shown in Fig. 9a. The strain energy of the bar is given by $\int$tde which the area of BCE of the load extension diagram. The final potential energy of the applied forces is equal to T(1 + e); Hence the area ABDE is -V and U - S = = -V which agrees with equation 54. A similar analysis for the pin-ended bar in compression shown in Fig. 9b also upholds equation 54.

The name stress energy is suggested because of certain complementary properties to strain energy. Both are forms of potential energy. If there is a change in the displacement of a system with no change in the forces then there will be no change in the stress energy but the strain energy may change. On the other hand if there is a change in the force system without any movement of the structure or mechanism then there will be no change in the strain energy but the stress energy may change. The theorem of the differential coefficient of the stress energy $\partial S/\partial F_j = \nabla_j$ which is applicable to nonlinear structures undergoing gross deformation may be contrasted with Castiglianos theorem of the differential coefficient of the internal work (Part I) $\partial U/\partial \nabla_j = P_j$ (in which ∇_j is the displacement corresponding to the force P_j), as this is also applicable to nonlinear structures undergoing gross deformation.

11. Liboves Complementary Energy Method

Libove [8] derives a form of complementary energy (called CE) which is just the negative of stress energy and exhibits similar differential propties. Libove also defines a theorem of stationary total complementary energy which is compared with the stationary part of the theorem of minimum total potential energy. However any implication that CE is a form of energy having a dual role to potential energy should be resisted, as CE is itself the negative of a form of potential energy.

The alternative use of stress energy is proposed in this paper because
a) there would be no confusion with Engesser's original complementary energy concept which still remains useful for situations in which structural deflections can be considered small.
b) stress energy is a positive form of potential energy as far as the system under consideration is concerned.
c) the title makes use of the duality in its relationship with strain energy.

Libove's theorem of stationary total complementary energy was stated to be weak because extraneous solution could sometimes be found in which the total complementary energy was stationary but which did not constitute compatible states for the system. Subsequent discussion on this subject [14, 15] centred around a problem which has been used in this paper and is illustrated by Fig. 4, and for which Libove records an extraneous solution when F = T and P = 0. Libove's original difficulty arose because the complementary energy was differentiated with respect to T and θ. Although θ is not a force, a variation of θ will result in a variation in the force system in all cases except the case where the force in AB is zero. Hence differentiation with respect to θ will give correct results except when the force AB is zero which is presicely the situation in which Libove obtained the extraneous solution.

If, in the same example, the total complementary energy is differentiated with respect to T and P then, with members all of area A and Youngs modulus E and writing p = P/EA, t = T/EA and f = F/EA, the following equations are obtained

$$p\left[\frac{2}{(p^2+(f-t)^2)^{\frac{1}{2}}}+2+1\right]=\sqrt{2}$$
$$(f-t)\left[\frac{2}{(p^2+(f-t)^2)^{\frac{1}{2}}}+2+1\right]=\sqrt{2}(1+f) \tag{55}$$

The correct solution in which

$$p=\frac{f-t}{1+f} \tag{56}$$

can be obtained by division of these equations.

However in the particular case where p = 0 and f-t = 0 the left hand sides of equations 55 are indeterminate and it may at first sight appear impossible to tell that there is not an extraneous solution.

The physical significance of this situation is that because no force is present in the member AB its angle is indeterminate. The compatibility conditions as specified by equation 21 can each be satisfied separately by choosing appropriate values of θ but the same value of θ will not satisfy both equations simultaneously. Examination of equations 55 will reveal that if p and f-t both tend to zero then

$$\frac{p}{(p^2+(f-t)^2)^{\frac{1}{2}}}=\frac{1}{\sqrt{2}}$$
$$\text{and}\ \frac{f-t}{(p^2+(f-t)^2)^{\frac{1}{2}}}=\frac{1+f}{\sqrt{2}} \tag{57}$$

Squaring and adding gives

$$1 = \frac{1}{2}\ [1 + (1 + f)^2] \qquad (58)$$

Equation 58 can only be satisfied in the trivial case where $f = 0$ and the impossible case where $f = -2$.

Thus, no extraneous solutions exist as long as the total complementary energy is differentiated with respect to changes in the force system only. This result also applies to differentiation of the stress energy and in the statement of the theorem it has been stated that differentiation is to be carried out with respect to a force.

12. *Charlton's Virtual Work Method*

Charlton [9] describes a method of obtaining the same results that can be obtained by using CE or stress energy. The technique adopted is to consider a virtual displacement of the structure or mechanism to one twice the size, and then to replace the forces in the resulting virtual work equations by their incremental values. The justification for the second step is explained in an appendix to Reference 3 added in 1964. In comparison with the stress energy method it is seen that the work required to move the force increments from an arbitrary origin to the structure will be equal to the work required to move the force increments from the structure to one twice the size.

13. *Conclusion*

Corresponding relationships between certain incremental equilibrium equations and compatibility equations can be derived for structures undergoing gross deformation. Investigation of these relationships leads to the development of a new principle of virtual work and a form of energy called stress energy. The concept of stress energy has been developed by considering the potential energy of the forces applied to the structure or mechanism and its relationship to Libove's from of complementary energy and Charlton's virtual work method has been discussed.

It is hoped that this paper helps to clarify the situation with regard to the use of virtual work methods and energy theorems as applied to structures undergoing gross deformation, whilst at the same time indicating a possible application to cable and suspension structures.

ACKNOWLEDGEMENTS

The author would like to thank Professor Charlton for valuable discussions on this subject.

References

1. Matheson, J. A. L.: Hyperstatic Structures. Butterworths Publications, London, 1959.
2. Argyris, J. H. and Kelsey, S.: Energy Theorems and Structural Analysis. Butterworths Publications, London, 1960.
3. Charlton, T. M.: Energy Theorems in Applied Statics. Blackie & Son. London, 1959.
4. Williams, D.: An Introduction to the Theory of Aircraft Structures. Edward Arnold (Publishers), London, 1960.

5. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw-Hill, New York, 1956.

6. Engesser, Fr.: Ueber Statische Unbesimmte Trager bei Beliebigem Formanderungs - Gesetze und uber den Satz von der Kleinsten Erganzungsarbeit. Z.d.Arch. - u.Ing. Ver.Z.Hannover, Vol. 35, col. 733, 1889.

7. Westergaard, H.M.: On the Method of Complementary Energy. Proc. ASCE. Vol. 67, pp. 199-227, 1941.

8. Libove, C.: Complementary Energy Method for Finite Deformation. Journal ASCE, Vol. 90, No. EM6, pp. 49-71, December 1964.

9. Charlton, T.M.: Strain Compatibility Conditions of Grossly Distorted Structures by Virtual Work. Civil Engineering, Vol. 58, pp. 325-326, 1963.

10. Levinson, M.: The Complementary Energy Theorem in Finite Elasticity. Journal of Applied Mechanics, Vol. 32, pp. 826-828, December 1965.

11. Jennings, A.: The Free Cable. The Engineer, Vol. 214, pp. 1111 and 1112, December, 1962.

12. O'Brien, T.: General Solution of Suspended Cable Problems. Journal ASCE, Vol. 93, No. ST. 1, pp 1-26, February, 1967.

13. Dugas, R.: A History of Mechanics. Routledge & Kegan Paul, London, 1955.

14. Jennings, A.: Discussion of Complementary Energy Method for Finite Deformation by C.Libove. Journal ASCE, Vol. 91, No. EM4, pp. 203-206, August, 1965.

15. Libove, C.: Closure of Discussion on Complementary Energy Method for Finite Deformation. Journal ASCE, Vol. 92, No. EM2, pp. 279-290, April, 1966.

[Received April 25, 1967]